

Violent Relaxation of Spherical Stellar Systems

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(Received ;accepted)

Abstract

Violent relaxation process of spherical stellar systems is examined by numerical simulations of shell model. The collapse of uniform density sphere both with and without external force is investigated. It is found that time variation of mean gravitational potential field induced by external force makes the system closer to Lynden-Bell distribution. Our result imply that systems which are gravitationary affected by other systems may approach the Lynden-Bell distribution.

Key words: Collisionless stellar dynamics; Elliptical galaxies; Violent relaxation

1 Introduction

Violent relaxation is a mechanism proposed to explain a similarity and smoothness of the observed light distribution of elliptical galaxies. This relaxation process is caused by explicitly time-dependant mean gravitational potential field. Its typical timescale is much shorter than two-body relaxation time and comparable with dynamical time.

According to Lynden-Bell (1967), it is expected that the coarse-grained entropy of systems becomes maximum through violent relaxation in ideal cases and that Lynden-Bell distribution can be established. The Lynden-Bell distribution is given by

$$\bar{f}(\varepsilon) = \eta \frac{\exp\{-\beta(\varepsilon - \mu)\}}{1 + \exp\{-\beta(\varepsilon - \mu)\}}, \quad (1)$$

where $\bar{f}(\varepsilon)$ is coarse-grained distribution function, ε is energy per unit mass, and η, β, μ are constants. In almost all cases, degeneracy can be neglected ($\bar{f}(\varepsilon) \ll \eta$), so Lynden-Bell distribution is approximated by the Maxwell-Boltzmann distribution:

$$\bar{f}(\varepsilon) = A \exp(-\beta\varepsilon), \quad (2)$$

where $A = \eta \exp(\beta\mu) = \text{constant}$. However, in numerical simulations so far of one dimensional gravitational N-body systems (sheet systems), they did not reach Lynden-bell distribution through violent relaxation (Cuperman et al. 1969; Hohl and Campbell 1968; Goldstein et al. 1969; Yamashiro

et al. 1992). It remains unclear why Lynden-Bell distribution cannot be obtained in these simulations. Moreover it has not been fully investigated what distributions are obtained in other gravitational N-body systems (shell systems, particle systems) through violent relaxation.

It is widely believed that the Lynden-Bell distribution is not realized in N-body simulations (e.g., Funato et al. 1992a, 1992b). One of the reasons is that the fluctuations in the gravitational potential decays before the relaxation process is completed. To investigate if this is true, we add artificial time (and space) varying potentials and see if the Lynden-Bell distribution is established. As a first step, we adopt simple spherical symmetric models since spherical symmetric models are more realistic than sheet models. More realistic models will be studied in subsequent papers.

2 Numerical Method

In our simulations, shell models (Hénon 1964) are adapted. In spherical symmetric stellar systems, distribution function can be expressed as $f(r, u, v, t)$, where r the radial distance, u the radial velocity, v the tangential velocity, and t is the time, respectively. So distribution function is represented by N points in (r, u, v) space and each point evolves according to the equations of motion.

$$\frac{dr_i}{dt} = u_i, \quad (3)$$

$$\frac{du_i}{dt} = \frac{A_i^2}{r_i^3} - \frac{GM_i}{r_i^2} - \frac{\partial \phi_{ex}(r, t)}{\partial r}, \quad (4)$$

where G the gravitational constant, $A_i = r_i v_i = \text{constant}$ (with respect to time) is the angular momentum of each shell, M_i the mass interior to r_i , and $\phi_{ex}(r, t)$ is the potential of external force.

We set $G = M(\text{total mass}) = 1$ and $N = 10000$. And each shell has the same mass ($= 1/N$). The integration is carried out using a leap-frog method.

When we set $\phi_{ex}(r, t) = 0$, we set integration time-step small enough for the maximum error of total energy to be of the order of 0.01% after $100t_{cr}$. t_{cr} is the crossing time which is typical timescale for shells to cross typical radius of the system. In our simulations, $t_{cr} \sim 1$ in this system of units. When $\phi_{ex}(r, t) \neq 0$, we set integration time-step same as the model whose initial condition is common.

3 Models

We calculate collapses of uniform density sphere whose initial virial ratio is 0.5 and 0.1. In some models external gravitational potential field are added after systems are settled down to study equilibrium states.

Initial densities and velocity distributions are common in all models.

- $\rho(r) = \text{const.}$ ($0 \leq r \leq 2$).
- Both u and v obey isotropic Maxwell distribution.

For simplicity, we assume that both time and spatial dependence of external potential field is sinusoidal. We set ϕ_{ex} as follows,

$$\phi_{ex}(r, t) = \begin{cases} A \sin(2\pi t/P) \sin(2\pi r/\lambda) & (20 \leq t \leq 20 + 5P) \\ 0 & (\text{other}) \end{cases} \quad (5)$$

In (5) P and λ are parameters. We set external potential to survive for much shorter time than two-body relaxation time which is comparable to Nt_{cr} (see appendix 1). We set typical intensity of external potential comparable to that of inertial potential. In our simulations, we set $A=0.5$ when initial virial ratio is 0.5, and $A=1.36$ when initial virial ratio 0.1.

In table 1 the parameters for each models are listed. In this table both P and λ are normalized in the units of \bar{R} and \bar{t} , which are typical lengthscale and time scale of the post collapsed system. In models whose initial virial ratio is 0.5, $\bar{R} = 1.34$ and $\bar{t} = 1.53$, and in models whose initial virial ratio is 0.1, $\bar{R} = 0.494$ and $\bar{t} = 0.345$ (see appendix 2). Models are summarized in table 1.

4 Results

In models A-0 and B-0, the system approached stationary states within a few crossing times and variation of virial ratio and variation of distribution function was damped (figure 1 and figure 2). In figure 2, the mean value of the virial ratio at the stationary stages is greater than 1.0 (about 1.35) because the existence of shells which have positive energies cannot be negligible.

Figure 1

Figure 2

In other models, the systems also approached stationary states within a few crossing time after variation of external potential stopped. In table 2 ratio of mass whose energy is more than zero in stationary states is listed in each model.

In figures 3, figures 4 and figures 5, distribution function of stationary states of model A-0, model B-0 and model A-1 are shown by the solid lines. We derive distribution function from the simulation data as follows. We divide energy space between $\varepsilon = \varepsilon_{min}$ and $\varepsilon = 0$ into 100 bins, where ε_{min} is the minimum value of the potential energy of the system. Therefor the i -th bin corresponds to the energy range between $\varepsilon_i - \Delta\varepsilon/2$ and $\varepsilon_i + \Delta\varepsilon/2$, where $\varepsilon_i = \varepsilon_{min} + (i - 1/2)\Delta\varepsilon$ and $\Delta\varepsilon = -\varepsilon_{min}/100$. Then we count the number of shells contained the i -th bin and it is divided by the phase space volume which corresponds to the energy range to get the $\bar{f}(\varepsilon_i)$. Anisotropy of velocity distribution cannot harm this procedure because energy of each shell doesn't depend on the direction of velocity but only on the absolute value.

Figure 3

Figure 4

Figure 5

We estimate the deviation from Lynden-Bell distribution of each model as follows. Unfortunately, spherical stellar systems with Maxwell distribution (isothermal distribution) have infinite mass. So the distribution expected after violent relaxation in real spherical stellar systems is not Maxwellian itself, but that of the King model which has 'cut-off' at higher energy end.

So distribution functions in stationary states are fitted by the form,

$$\bar{f}(\varepsilon) = \rho_1 (2\pi\sigma^2)^{-3/2} \left\{ \exp\left(\frac{-\varepsilon + \Phi_0}{\sigma^2}\right) - 1 \right\}, \quad (6)$$

where, ρ_1 , σ , Φ_0 are fitting parameters. In figure 3, figure 4 and figure 5, distribution functions of the best fit King-models are showed by the short dash lines.

Then we define D^2 as follows and use it for the estimation of the deviation from Lynden-Bell distribution.

$$D^2 \equiv \left(\sum_{i=1}^{N_b} n_i \right)^{-1} \sum_{i=1}^{N_b} n_i \left| \frac{\bar{f}_{data}(\varepsilon_i) - \bar{f}_{fit}(\varepsilon_i)}{\bar{f}_{fit}(\varepsilon_i)} \right|^2, \quad (7)$$

where, \bar{f}_{data} is the distribution function from simulation data, \bar{f}_{fit} is the best fit function of the data in the form of (6), N_b is the number of bins (in this paper $N_b = 100$) and n_i is the number of shells contained in the i -th bin. Fluctuations from the finiteness of shell number are proportional to square root of shell number. Therefore, the summation is carried out weighted with n_i in the numerator of equation (7).

In table 3, fitting parameters (ρ_1, σ^2, Φ_0), c , which is the concentration parameter of the corresponding King model obtained from the fitting result and D^2 of each model are listed. Compared between Model A-0 and B-0, both of which are without external force, D^2 of B-0 is smaller than that of A-0. Initial condition of model B-0 is farther from dynamical equilibrium state than that of A-0, so time variation of gravitational potential is larger and relaxation is more effective. In all models whose initial virial ratios are

0.5 with external force (from A-1 to A-9), D^2 becomes smaller than that of model A-0 which has the same initial condition but without external force. The similar tendency can be seen in almost models whose initial virial ratio is 0.1 (from B-0 to B-1) except B-4 and B-7.

5 Discussion

We examined the violent relaxation process of spherical stellar systems and the influence on it of time variation of external gravitational potential fields by using shell models. It is found that time variation of the mean potential field mainly caused by external gravitational field makes distribution function closer to Maxwellian. Especially in some models (A1, A4, A7, B8, B9) the deviations from the King model become quite small. Though it is not clear why the deviations are small in these models, it shows that the violent relaxation is quite efficient under some conditions.

So, if time variation of mean gravitational potential fields continues for a long enough time, systems can reach Lynden-Bell distribution.

It is rare that no gravitational influence of other galaxies is exercised on galaxies. Especially in galaxies in clusters, this effect plays an important role on their dynamical evolution. Though it is not clear the real gravitational perturbations are how much effective to cause the violent relaxation, there are possibilities that the distribution function of the real elliptical galaxies become close to the King models by the external forces.

We would like to thank T.Hayashi for his contribution to developing numerical code and M. Shimada for helpful discussions. Numerical computations in this work were carried out with workstations at Yukawa Institute for Theoretical Physics. This work is in part supported by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists (M.T.).

Appendix 1. Estimation of Two-Body Relaxation Time of Shell Systems

We assume that there are N shells which have the same mass of m . Then the mean energy of each shell $\bar{\varepsilon}$ is as follows from the Virial theorem.

$$\bar{\varepsilon} \sim -\frac{GNm^2}{\bar{R}}, \quad (\text{A1.1})$$

where G is the gravitational constant and \bar{R} is a typical radius of the system.

And the mean energy change of each shell in one shell-crossing-event is

$$|\bar{\Delta\varepsilon}| \sim \frac{Gm^2}{\bar{R}}. \quad (\text{A1.2})$$

Relaxation process can be regarded as one-dimensional random walks in the energy space whose one step-size is equal to $|\bar{\Delta\varepsilon}|$. Since the system can be regarded as relaxed when the variance of displacement in the energy space becomes comparable to the square of typical energy of each shell, number of crossing-event required for relaxation, n_{rel} , is calculated as follows.

$$\bar{\varepsilon}^2 \sim n_{rel} |\bar{\Delta\varepsilon}|^2, \quad (\text{A1.3})$$

$$n_{rel} \sim \left(\frac{\bar{\varepsilon}}{\Delta\varepsilon}\right)^2 \sim N^2. \quad (\text{A1.4})$$

Then relaxation time of shell systems, t_{rel} , is

$$t_{rel} \sim n_{rel} \frac{\bar{r}}{\bar{v}} \sim N \frac{\bar{R}}{\bar{v}} \sim N t_{cr}, \quad (\text{A1.5})$$

where $\bar{r} = \bar{R}/N$ is the mean separation between shells and \bar{v} is the mean velocity of shells.

Appendix 2. Changes of Length and Time Scale through Collapse

We assume that the initial total mass is M , the initial typical length scale is R_0 , and that the initial velocity dispersion is v_0^2 . Then total kinetic energy, T_0 , total potential energy, W_0 , and virial ratio, V_0 , of initial states are,

$$T_0 = \frac{1}{2} M v_0^2, \quad (\text{A2.1})$$

$$W_0 = -\frac{GM^2}{R_0}, \quad (\text{A2.2})$$

$$V_0 = \left| \frac{2T_0}{W_0} \right| = \frac{v_0^2 R_0}{GM}, \quad (\text{A2.3})$$

where G is the gravitational constant.

When the system reached virial equilibrium after the collapse, we let the typical length scale R , velocity dispersion v^2 , and the total mass $M(1-x)$, where x is the escape rate of mass. Then total kinetic energy, T , total potential energy, W , and virial ratio, V , of the bounded portion of the system

are,

$$T = \frac{1}{2}M(1-x)v^2, \quad (\text{A2.4})$$

$$W = -\frac{GM^2}{R}(1-x)^2, \quad (\text{A2.5})$$

$$V = 1 = \frac{v^2 R}{GM(1-x)}. \quad (\text{A2.6})$$

If we let the energy which escaped mass possess ΔE , according to energy conservation law,

$$T_0 + W_0 = T + W + \Delta E, \quad (\text{A2.7})$$

then,

$$v^2 = \frac{GM}{R_0}(2 - Vr_0)(1-x)^{-1}\left(1 + \frac{\Delta E}{|E_0|}\right) \quad (\text{A2.8})$$

$$= v_0^2\left(-1 + \frac{2}{Vr_0}\right)(1-x)^{-1}\left(1 + \frac{\Delta E}{|E_0|}\right), \quad (\text{A2.9})$$

$$R = R_0(2 - Vr_0)^{-1}(1-x)^2\left(1 + \frac{\Delta E}{|E_0|}\right)^{-1}, \quad (\text{A2.10})$$

$$t = \frac{1}{\sqrt{GM}}\left(\frac{R_0}{2 - Vr_0}\right)^{2/3}(1-x)^{5/2}\left(1 + \frac{\Delta E}{|E_0|}\right)^{-3/2} \quad (\text{A2.11})$$

$$= t_0 \frac{Vr_0^{1/2}}{(2 - Vr_0)^{3/2}}(1-x)^{5/2}\left(1 + \frac{\Delta E}{|E_0|}\right)^{-3/2}, \quad (\text{A2.12})$$

where $|E_0| = GM^2(1 - Vr_0/2)/R_0$ is the absolute value of initial total energy of the system, $t = R/v$ is the typical timescale of the system after the system has collapsed, and $t_0 = R_0/v_0$ is that of initial state.

In our simulation, when $V_0 = 0.5$, x and $\Delta E/|E_0|$ is much smaller than unity and can be negligible. On the other hand, when $V_0 = 0.1$, $x = 0.32$ and

$\Delta E/|E_0| = 0.45$. With our unit ($G = M = 1$) and the initial conditions described in section 3, when $V_0 = 0.5$, $R = 1.34$ and $t = 1.53$ from (A2.10) and (A2.11). When $V_0 = 0.1$, $R = 0.494$ and $t = 0.345$.

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Table 1. Parameters of Models

Model	Initial V.R.	External P.F.	P/\bar{t}	λ/R
A-0	0.5	No	–	–
A-1	0.5	Yes	0.5	0.5
A-2	0.5	Yes	0.5	1.0
A-3	0.5	Yes	0.5	2.0
A-4	0.5	Yes	1.0	0.5
A-5	0.5	Yes	1.0	1.0
A-6	0.5	Yes	1.0	2.0
A-7	0.5	Yes	2.0	0.5
A-8	0.5	Yes	2.0	1.0
A-9	0.5	Yes	2.0	2.0
B-0	0.1	No	–	–
B-1	0.1	Yes	0.5	0.5
B-2	0.1	Yes	0.5	1.0
B-3	0.1	Yes	0.5	2.0
B-4	0.1	Yes	1.0	0.5
B-5	0.1	Yes	1.0	1.0
B-6	0.1	Yes	1.0	2.0
B-7	0.1	Yes	2.0	0.5
B-8	0.1	Yes	2.0	1.0
B-9	0.1	Yes	2.0	2.0

Table 2. Ratio of mass whose energy is more than zero in stationary states.

Model	Rate	Model	Rate
A-0	0.0003	B-0	0.33
A-1	0.78	B-1	0.85
A-2	0.11	B-2	0.39
A-3	0.0029	B-3	0.31
A-4	0.41	B-4	0.60
A-5	0.78	B-5	0.82
A-6	0.20	B-6	0.43
A-7	0.18	B-7	0.42
A-8	0.53	B-8	0.64
A-9	0.75	B-9	0.80

Table 3. Fitting parameters (ρ_1 , σ^2 , Φ_0) and concentration parameter, c , and D^2 of each model

Model	ρ_1	σ^2	Φ_0	c	D^2
A-0	2.43×10^{-4}	0.181	-2.25×10^{-2}	1.52	1.26
A-1	6.31×10^{-9}	9.80×10^{-3}	-1.56×10^{-3}	2.44	0.360
A-2	6.05×10^{-5}	0.102	-1.29×10^{-2}	1.54	0.725
A-3	2.13×10^{-4}	0.168	-2.10×10^{-2}	1.51	0.954
A-4	4.32×10^{-7}	2.71×10^{-2}	-4.18×10^{-3}	2.05	0.427
A-5	5.71×10^{-9}	9.80×10^{-3}	-1.67×10^{-3}	2.33	0.588
A-6	2.57×10^{-6}	4.27×10^{-2}	-6.15×10^{-3}	1.86	1.19
A-7	1.60×10^{-5}	7.40×10^{-2}	-8.95×10^{-3}	1.67	0.363
A-8	4.81×10^{-6}	5.32×10^{-2}	-6.62×10^{-3}	1.61	0.506
A-9	2.22×10^{-8}	1.33×10^{-2}	-2.11×10^{-3}	2.13	0.852
B-0	1.80×10^{-2}	0.541	-6.48×10^{-2}	1.46	0.923
B-1	1.20×10^{-9}	7.19×10^{-3}	-1.52×10^{-3}	3.01	0.633
B-2	2.12×10^{-3}	0.227	-2.67×10^{-2}	1.45	0.720
B-3	1.50×10^{-2}	0.485	-5.60×10^{-2}	1.41	0.726
B-4	1.55×10^{-5}	5.32×10^{-2}	-8.65×10^{-3}	2.11	1.04
B-5	9.33×10^{-9}	1.04×10^{-2}	-2.31×10^{-3}	3.01	0.534
B-6	6.17×10^{-4}	0.149	-1.82×10^{-2}	1.49	0.779
B-7	4.93×10^{-4}	0.151	-1.81×10^{-2}	1.67	1.29
B-8	9.48×10^{-7}	3.32×10^{-2}	-6.35×10^{-3}	2.60	0.387
B-9	2.12×10^{-8}	1.21×10^{-2}	-2.48×10^{-3}	2.78	0.250

Figure captions

Fig. 1. Time variation of virial ratio of model A-0. The system approached stationary state within a few crossing times and variation of virial ratio was damped.

Fig. 2. Same as figure 1, but for model B-0. the mean of virial ratio of stationary state is greater than 1.0 (about 1.35) because the existence of shells which have energy of more than zero cannot be negligible.

Fig. 3. Distribution function of stationary states of model A-0 (by the solid line) and that of the best fit King-model (by the short dash line).

Fig. 4. Same as figure 3, but for model B-0.

Fig. 5. Same as figure 3, but for model A-1. $\bar{f}(\varepsilon)$ obeys Maxwellian. The deviation from Maxwellian of lower energy region is due to the statistical fluctuation because number of shells in this energy range is rather small.









